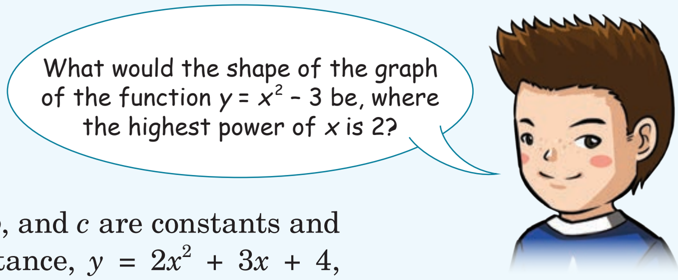
8.2 *Graphs of Quadratic Functions*

In an earlier section, we have learned that the graph of the linear function $y = mx + b$, where the highest power of x is 1, is a straight line.



In general, the function $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$, is called a **quadratic function**. For instance, $y = 2x^2 + 3x + 4$, $y = x^2 - 3$, and $y = -x^2 - 6x + 1$ are quadratic functions y of x.

Let us first explore the graph of the quadratic function $y = ax^2 + bx + c$, where the coefficient of x^2 is positive, i.e. $a > 0$. The graph of a quadratic function is called a **quadratic graph**.

A Graph of $y = ax^2 + bx + c$, where $a > 0$

The simplest quadratic function is $y = x^2$. We can draw its graph as follows.

STEP 1 Set up a table of values.

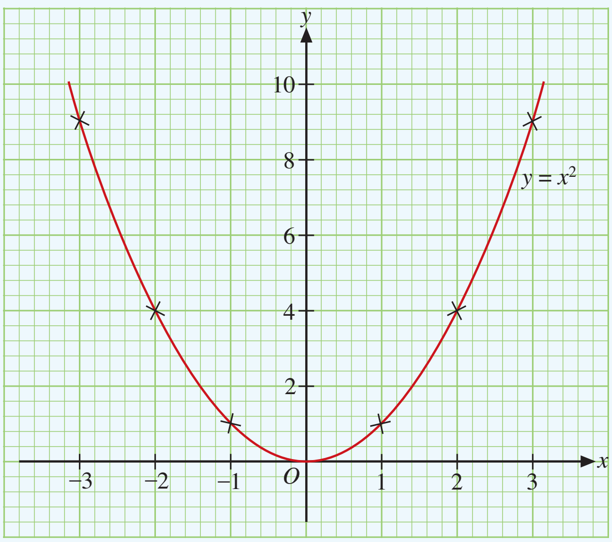
x	–3	–2	–1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

REMARKS

When $a = 1$, $b = 0$, and $c = 0$, $y = ax^2 + bx + c$ becomes $y = x^2$.

STEP 2 Plot the points on a Cartesian plane.

STEP 3 Join the points to form a smooth curve. The curve is the graph of $y = x^2$.



The graph of $y = x^2$ is a curve that is symmetrical about the y-axis. We say that it opens upwards. Do all quadratic graphs have the same shape as the above curve? Let us draw more quadratic graphs and investigate their properties.

CLASS ACTIVITY 5

Objective: To investigate the properties of the graph of a quadratic function $y = ax^2 + bx + c$ for $a > 0$.

Questions

1. (a) Copy and complete the following tables.

(a) (i)

x	–4	–3	–2	–1	0	2	3	4
$y = \frac{1}{2}x^2$	9	4	1	0	1	4	9	

(ii)

x	–2	–1.5	–1	0	1	1.5	2
$y = 2x^2$	9	4	1	0	1	4	9

(iii)

x	–2	–1.5	–1	0	1	1.5	2
$y = 2x^2 + 1$							

(iv)

x	–2	–1.5	–1	0	1	1.5	2
$y = 2x^2 - 3$							

(v)

x	–3	–2	–1	0	1	2	3
$y = x^2$							

(vi)

x	–4	–3	–2	–1	0	1	2
$y = (x + 1)^2$							

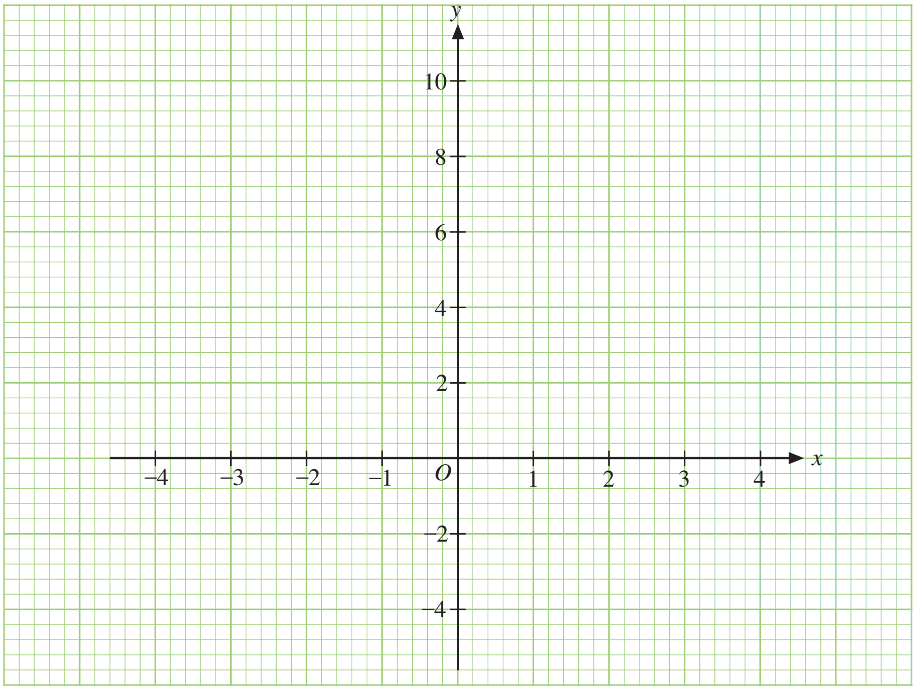
(vii)

x	–2	–1	0	1	2	3	4
$y = (x - 1)^2$							

(viii)

x	–2	–1	0	1	2	3	4
$y = (x - 1)^2 - 2$							

(b) Draw the graphs of the equations in **(i)**, **(ii)**, **(iii)**, and **(iv)** on a sheet of graph paper using the same scales as shown below.



(c) Draw the graphs of the equations in **(v)**, **(vi)**, **(vii)**, and **(viii)** on another sheet of graph paper using the same scales as that in **(b)**.

(d) **(i)** Describe the graphs in **(b)** and **(c)**.

(ii) Write down the coordinates of the lowest point of each graph.

(iii) Which graph has the widest shape?

(iv) At how many points does each graph meet the x-axis?

(v) At how many points does each graph meet the y-axis?

(vi) Study the shapes of the graphs. Are they symmetrical? If they are, state the equation of the line about which each graph is symmetrical.

In Class Activity 5, we generalized the equation $y = x^2$ in three ways. One was to multiply x^2 by a positive number such that $y = ax^2$. The numbers 2 and $\frac{1}{2}$ were used in the activity but it could be other positive numbers.

When the number is larger than 1, the graph becomes narrower. The graph becomes broader when the number is between 0 and 1.

Next, we added or subtracted a number such that $y = ax^2 + c$, as in (iii) and (iv). This either raised or lowered the graph $y = ax^2$ in the Cartesian plane.

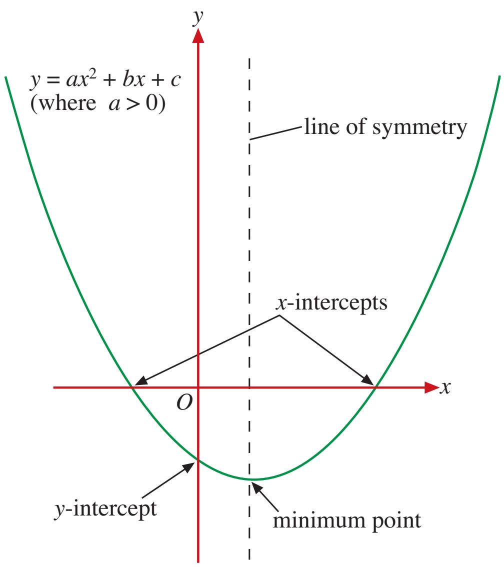
Finally, we added or subtracted a constant from the variable x, such that $y = (x + 1)^2$ and $y = (x - 1)^2$, as in (vi) and (vii). The original graph $y = x^2$ is shifted to the left or right along the x-axis. In all cases, the shape and size of the original graph remains the same. These three operations can be combined to get the general function $y = ax^2 + bx + c$.

DISCUSS

What are the expanded forms of the quadratic functions $y = (x + 1)^2$, $y = (x - 1)^2$ and $y = (x - 1)^2 - 2$?

The graphs of quadratic functions $y = ax^2 + bx + c$, for $a > 0$, have the following properties which were illustrated in Class Activity 5.

1. When $a > 0$, the graph opens upward. It is sometimes called a **parabola**.
2. The graph has a lowest point. This point is called the **minimum point** of the parabola.
3. The vertical line through the minimum point is the **line of symmetry** of the graph.
4. The smaller the numerical value of a, the wider the graph opens upward.
5. The graph may meet the x-axis at 0, 1 or 2 points. However, it meets the y-axis at only 1 point.



MATH WEB

You can look up 'parabola' on the Internet to read about some of its properties. Most of these will be treated in later courses.

REMARKS

The x-coordinate of a point of intersection of a graph and the x-axis is called an ***x*-intercept** of the graph.

Example 4

A horizontal bridge is 20 m long and it is supported by a suspension cable. The height, y m, of the cable over the bridge at a distance, x m, from one end of the bridge is given by

$$y = 0.03x^2 - 0.6x + 5, \quad \text{for } 0 \leqslant x \leqslant 20.$$

(a) Draw the graph of $y = 0.03x^2 - 0.6x + 5$ for $0 \leqslant x \leqslant 20$.
(b) Find the height of the cable at one end.
(c) State the line of symmetry of the graph.
(d) Find the minimum height of the cable above the bridge.

Solution

(a) We set up a table of values for the function.

$y = 0.03x^2 - 0.6x + 5$

x	0	4	8	10	12	16	20
y	5	3.08	2.12	2	2.12	3.08	5

4. **(a)** Copy and complete the following table.

x	–5	–4	–3	–2	–1	0	1
$y = -3x^2 - 18x - 32$							

(b) Taking 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graph of $y = -3x^2 - 18x - 32$ for $-5 \leqslant x \leqslant 1$.

(c) State the line of symmetry and the maximum point of the graph.

(d) Find the y-intercept of the graph.

(e) At how many points does the graph cut the x-axis?

FURTHER PRACTICE

5. Draw the graph of each of the following equations for the indicated values of x.

(a) $y = \frac{1}{4}x^2 + x$ for $-5 \leq x \leq 1$

(b) $y = 2x^2 - 6x + 7$ for $-1 \leqslant x \leqslant 4$

(c) $y = 6 - x - x^2$ for $-4 \leqslant x \leqslant 3$

(d) $y = -2x^2 + 3x - 3$ for $-2 \leqslant x \leqslant 3$

For each graph, state its

(i) line of symmetry,

(ii) minimum or maximum point,

(iii) x-intercept(s),

(iv) y-intercept.

6. For the graph of equation **5(d)**, are the points $\left(\frac{1}{2}, -1\right)$ and $\left(\frac{3}{2}, -3\right)$ on or above or below the graph of $y = -2x^2 + 3x - 3$?

MATH@WORK

7. A store has a rectangular signboard of dimensions 7 m by 3 m. The store manager changes the dimensions by decreasing the length by x m, while increasing the width by x m. The area of the signboard is A m^2 after the changes.

(a) Express A in terms of x.

(b) Draw the graph of A against x for $0 \leqslant x \leqslant 7$.

(c) Find the value of x that gives the greatest area of the signboard. What is this area?

8. The height, h m, of a golf ball at time, t s, from the ground is given by $h = -5t^2 + 15t$.

(a) Find the time of flight of the golf ball.

(b) Draw the graph of $h = -5t^2 + 15t$ during the time of flight.

(c) Find the maximum height of the golf ball from the ground.

(d) When is the golf ball 8 m above the ground?

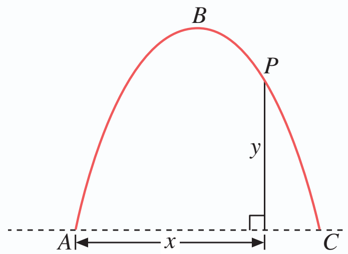
9. An arch ABC is a parabola with equation $y = -3x^2 + 6x$ for $0 \leqslant x \leqslant 2$, where x is the horizontal distance and y is the vertical distance, in meters, of a point, P, on the arch from the foot, A, of the arch (refer to the diagram).

(a) Draw the graph of the arch, taking 2 cm to represent 0.5 m on both axes.

(b) How wide is the bottom, AC, of the arch?

(c) Find the maximum height of the arch.

(d) State the line of symmetry of the arch.



10. When x chairs are made in a day, the cost \$$y$ for each chair is given by

$$y = x^2 - 20x + 150 \quad \text{for } 0 \leqslant x \leqslant 15.$$

(a) Draw the graph of $y = x^2 - 20x + 150$ for $0 \leqslant x \leqslant 15$.

(b) How many chairs have to be made in a day so that the cost per chair is a minimum?

(c) What are the possible numbers of chairs to be made in a day so as to keep the cost per chair less than \$60?

BRAIN WORKS

11. Quadratic graphs can be drawn using the command **Graph-Plot New Function** in Sketchpad software. Use the software to explore the following.
   (a) Draw the graphs of $y = ax^2$ for several values of a $\left(\text{such as } a = -3, -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2, 3\right)$ on the same diagram. Discuss the effect of a on the shapes and positions of the graphs.
   (b) Draw the graphs of $y = x^2 + bx + 4$ for several values of b on the same diagram. Discuss the effect of b on the shapes and positions of the graphs.
   (c) Draw the graphs of $y = -x^2 - 2x + c$ for several values of c on the same diagram. Discuss the effect of c on the shapes and positions of the graphs.

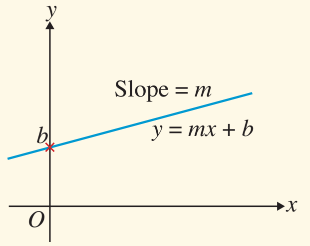
IN A NUTSHELL

Idea of Functions

- y is a function of x if for each x, there is exactly one value of y which is determined by x.
- A function can be represented by
  (i) a table of values,
  (ii) ordered pairs,
  (iii) a graph, or
  (iv) an equation.

Linear Function

- A linear function y of x is a function of the form $y = mx + b$, where m and b are constants.
- The graph of a linear function $y = mx + b$ is a straight line where m is the slope and b is the y-intercept of the line.



Properties of the Graph of the Quadratic Function $y = ax^2 + bx + c$, where $a \neq 0$

- It is in the shape of a parabola.
- It opens upward and has a minimum point when $a > 0$. It opens downward and has a maximum point when $a < 0$.
- It has a vertical line of symmetry through its minimum point or maximum point.
- It meets the x-axis at zero, one or two points.
- It meets the y-axis at only one point.

REVIEW EXERCISE 8

1. A linear function y of x is given by the following table.

x	1	3	5	7	9
y	1	5	9	13	17

(a) Represent the function by
  (i) ordered pairs,
  (ii) an equation.
(b) State the slope and the y-intercept of the linear graph as defined by the equation.
(c) Draw the graph of the function for $-1 \leqslant x \leqslant 9$.

2. The table shows the distance y miles a cyclist covered in x minutes.

Time (min)	Distance (miles)
30	8
60	16
90	24
120	32

(a) Draw a graph to represent the given data.
(b) Find the rate of change of this function.
(c) What does the rate of change represent?
(d) Express y as a function of x.

3. The number of minutes, x, provided under different cell phone plans and the costs, y, are shown in the table below.

Minutes	300	500	700	900	1100
Cost ($)	24	31	38	45	52

(a) Draw a graph to represent the given data.
(b) Find the rate of change of this function and explain what it represents.
(c) Find the initial value of this function. What does it represent?
(d) Express y as a function of x.

4. The value of a new car is $80,000. In 5 years' time, the car will be worth $20,000. Assume that the value of the car decreases at a constant rate of $\$m$ per year, and the value of the car is $\$y$ after x years.
(a) Find the value of m.
(b) Copy and complete the following table.

x	0	1	2	3	4	5
y						

(c) Express y as a function of x.
(d) Draw the graph of the function.

5. A pottery will supply 100 pots if the price of each pot is $80. It will supply 175 such pots if the price is $110. Assume that the supply quantity (y) is related to the price ($\$x$) of each pot by the function $y = mx + b$, where m and b are constants.
(a) Find the values of m and b.
(b) Draw the graph of the function $y = mx + b$ for $0 \leqslant x \leqslant 150$.
(c) What is the minimum price of each pot at which the pottery is willing to supply the pots?

6. **(a)** Draw the graph of $y = x^2 - 6x + 7$ for $0 \leqslant x \leqslant 6$.
(b) State the coordinates of the minimum point of the graph.
(c) Find the x-intercepts of the graph, rounding your values to 1 decimal place.

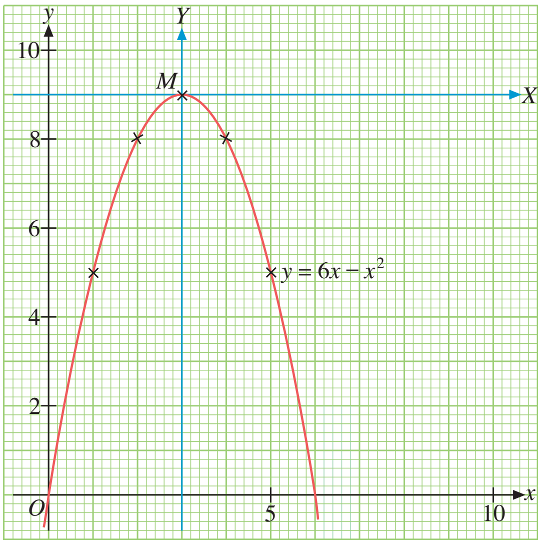
7. **(a)** Draw the graph of $y = 2x^2 + 6x + 5$ for $-5 \leqslant x \leqslant 1$.
(b) State the line of symmetry and the minimum point of the graph.
(c) Determine if each of the points $(-2.5, 2.5)$ and $(0.5, 9.5)$ lies on the graph of $y = 2x^2 + 6y + 5$, or above or below this graph.

EXTEND YOUR LEARNING CURVE

Translation of Axes

The diagram shows the graph of the quadratic function $y = 6x - x^2$. The point M is the maximum point of the graph.

(a) State the coordinates of M.



Suppose that we draw a horizontal axis, MX, and a new vertical axis, MY, at the point M.

(b) With reference to these new axes, MX and MY,

(i) state the coordinates of the point M,

(ii) state the coordinates of the point O,

(iii) do you think the equation of the graph still takes the form $Y = aX^2 + bX + c$? If not, what form do you think the equation of the graph should be?

(iv) find the equation of the graph.

WRITE IN YOUR JOURNAL

Describe the properties of

(a) a linear graph $y = mx + b$,

(b) a quadratic graph $y = ax^2 + bx + c$

that you have learned in this chapter.