LESSON 15

Introduction to Polygons

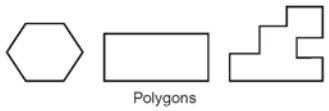
Warm Up

1. (2) **Vocabulary** A and B are the _____ of $\overline{AB}$.
2. (3) How many endpoints does a ray have?
3. (13) Classify each of the triangles in the diagram by both sides and angles.

New Concepts

A **polygon** is a closed plane figure formed by three or more segments. Each segment intersects exactly two other segments only at their endpoints. No two segments with a common endpoint are collinear.

The segments that form a polygon are called its **sides**. A **vertex of a polygon** is the intersection of two of its sides.



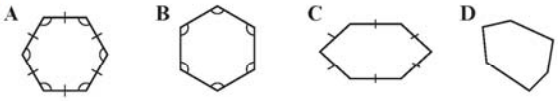
Polygons

Not Polygons

Hint

Equilateral and equiangular polygons have the same traits as equilateral and equiangular triangles, as introduced in lesson 13.

An **equiangular polygon** is a polygon in which all angles are congruent. An **equilateral polygon** is a polygon in which all sides are congruent. If a polygon is both equiangular and equilateral, then it is called a **regular polygon**. If a polygon is not equiangular and equilateral, then it is called an **irregular polygon**.



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In the diagram, polygons A and B are equiangular. Polygons A and C are equilateral. Since polygon A is both equiangular and equilateral, it is a regular polygon. Polygons B, C and D are all irregular.

Math Language

An ***n*-gon** is a polygon with *n* sides. Problems may refer to *n*-gons when the number of sides of a polygon is not known, or when a solution is desired for all possible polygons.

Polygons are named by the number of sides they have. The chart below shows some common polygons and their names.

Name	Sides	Regular Polygon	Irregular Polygon
Triangle	3		
Quadrilateral	4		
Pentagon	5		
Hexagon	6		
Heptagon	7		
Octagon	8		
Nonagon	9		
Decagon	10		
Hendecagon	11		
Dodecagon	12		

Example 1 **Classifying Polygons**

Classify each polygon. Determine whether it is equiangular, equilateral, regular, irregular, or more than one of these.

A 

B 

C 

D 

SOLUTION

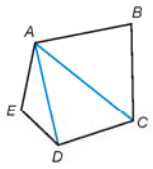
Polygon **A** has 5 sides, so it is a pentagon. It is equiangular but not equilateral, so it is irregular.

Polygon **B** has 7 sides, so it is a heptagon. It is equilateral and irregular.

Polygon **C** is a dodecagon. It is irregular.

Polygon **D** is a quadrilateral. It is equilateral and equiangular, so it is regular.

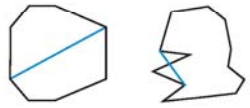
A **diagonal of a polygon** is a segment that connects two nonconsecutive vertices of a polygon. For example, pentagon $ABCDE$ has two diagonals, $\overline{AC}$ and $\overline{AD}$, from vertex A. Three other diagonals could be drawn: $\overline{BD}$, $\overline{BE}$, and $\overline{CE}$.



Math Language

The **exterior of a polygon** is the set of all points outside a polygon. The **interior of a polygon** is the set of all points inside a polygon.

Diagonals can help determine whether a polygon is concave or convex. In a **convex polygon**, every diagonal of the polygon lies inside it, except for the endpoints. In a **concave polygon**, at least one diagonal can be drawn so that part of the diagonal contains points in the exterior of the polygon.

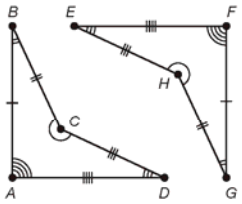


Convex polygon Concave polygon

If two polygons have the same size and shape, they are **congruent polygons**.

Example 2 **Identifying Polygon Properties**

a. Find a diagonal that contains points in the exterior of polygon $ABCD$.



SOLUTION

Diagonal $\overline{BD}$ lies outside polygon $ABCD$, except for its endpoints.

b. Determine whether polygon $EFGH$ is convex or concave. Explain.

SOLUTION

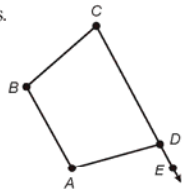
Diagonal $\overline{EG}$ contains points in the exterior of polygon $EFGH$. Therefore, polygon $EFGH$ is concave.

c. Are polygons $ABCD$ and $FGHE$ congruent? Justify your answer.

SOLUTION

Write a congruency statement for all corresponding sides and angles. Angle pairs $\angle A \cong \angle F$, $\angle B \cong \angle G$, $\angle C \cong \angle H$, and $\angle D \cong \angle E$. Sides $\overline{AB} \cong \overline{FG}$, $\overline{BC} \cong \overline{GH}$, $\overline{CD} \cong \overline{HE}$, and $\overline{DA} \cong \overline{EF}$. Therefore, $ABCD \cong FGHE$.

At each vertex of a polygon, there are two special angles. An **interior angle of a polygon** is an angle formed by two sides of a polygon with a common vertex. An **exterior angle of a polygon** is an angle formed by one side of a polygon and the extension of an adjacent side. In the diagram, $\angle CDA$ is an interior angle and $\angle ADE$ is an exterior angle.

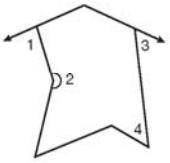


Math Reasoning

Generalize How many interior angles does a convex *n*-gon have? How many exterior angles does a convex *n*-gon have?

Example 3 Identifying Interior and Exterior Angles of Polygons

For each numbered angle in the polygon, determine whether it is an interior angle or an exterior angle.

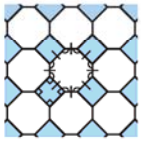


SOLUTION

Angles 2 and 4 are interior. Angles 1 and 3 are exterior.

Example 4 Application: Tile Patterns

This floor tile pattern uses polygonal tiles that fit together exactly.



a. Name the two types of polygons used in the pattern. Are they regular or irregular? Explain.

SOLUTION

Square and octagon; both types are regular, because they have all sides and all angles congruent, respectively.

b. Pick any pair of unshaded polygons. Are they congruent? Are they convex or concave? Explain.

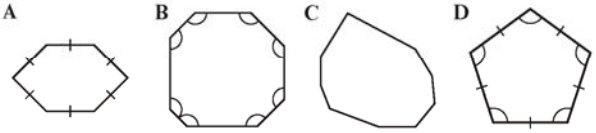
SOLUTION

All pairs of unshaded polygons are congruent, because corresponding sides and angles are congruent. Each unshaded polygon is convex, because none of the polygon's diagonals contain points in its exterior.

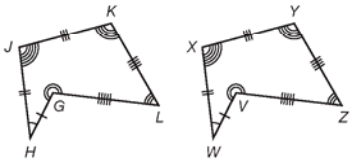
Lesson Practice

(Ex 1) **a.** Name each polygon. Determine whether it is equiangular, equilateral, regular, irregular, or more than one of these.

A B C D



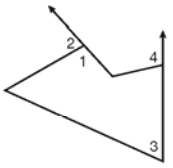
(Ex 2) **b.** Find a diagonal in polygon *GHJKL* that contains points in the exterior of the polygon.



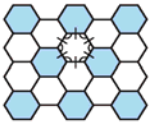
(Ex 2) **c.** Determine whether polygon *VWXYZ* is convex or concave. Explain.

(Ex 2) **d.** Are polygons *GHJKL* and *VWXYZ* congruent? Justify your answer.

(Ex 3) **e.** For each numbered angle in the polygon, determine whether it is an interior angle or an exterior angle.



(Ex 4) **f.** Name the type of polygon used in this pattern. Are the polygons regular or irregular? Explain.



(Ex 4) **g.** Pick any pair of polygons in this pattern. Are they congruent? Are they convex or concave? Explain.

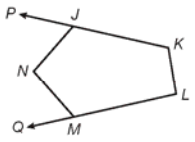
Practice Distributed and Integrated

(14) **1. Write** Explain why the pair of numbers 3 and 3 is a counterexample to this statement. *If the sum of two numbers is even, then both numbers are even.*

(5) **2.** **Agriculture** A farm is being divided so that each section of land has equal access to the canal running through the property for watering crops. If the road on the opposite side of the property runs parallel to the canal, explain how this can be done.

(8) **3.** **Wallpaper** A family wants to install wallpaper around the bottom half of a room. If the room has 12-foot tall ceilings and each wall is 14 feet long, calculate the area the wallpaper will cover.

(15) **4.** In polygon *JKLMN*, name each angle and identify it as an interior or exterior angle.



Write a conditional statement from each sentence.

(10) **5.** The absolute value of a number is a nonnegative number.

(10) **6.** A bilingual person speaks two languages.

(11) **7.** Determine the midpoint M of $\overline{XY}$ connecting $X(0, 2)$ and $Y(6, 1)$.

(14) **8. Model** Find a counterexample to this conjecture.
If two lines intersect, then any third coplanar line intersects both of them.

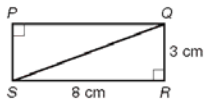
(7) **9.** Use inductive reasoning to determine the pattern in the following sequence:
2, 3, 5, 9, 17, 33, 65

(2) **10.** Identify the property that justifies this statement:
$a = b$ *and* $b = c$, *so* $a = c$.

11. (Inv 1) Name all the pairs of angles that are congruent when a transversal cuts a pair of parallel lines.

12. (9) Find the length of the segment connecting (1.3, 4.1) and (2.8, 6.1).

13. (13) **Verify** Rectangle $PQRS$ is divided into two triangles by diagonal $\overline{QS}$.



a. Determine the area of rectangle $PQRS$.

b. Given that $\triangle PQS$ and $\triangle QRS$ have equal areas, use your answer to part **a** to determine the area of $\triangle PQS$.

c. Verify that the formula for the area of a triangle gives the same answer as part **b** for the area of $\triangle PQS$.

14. (15) Classify this polygon. Is it equiangular? Is it equilateral? Is it regular?

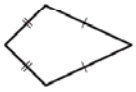
15. (12) **Home Renovation** Tasha is using two wooden rails to construct a pair of stair rails along the walls of a staircase. To make sure that the rails are parallel, she measures the acute angle each rail makes with the vertical edge of the wall at the base of the stairs.

a. What type of angles are these?

b. Tasha measures each angle to be 42°. Explain how she can make sure that the rails are parallel.

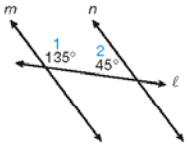
16. (14) **Verify** Confirm that this quadrilateral is a counterexample to the conjecture.

If a quadrilateral has two pairs of congruent sides, then both pairs of opposite sides are parallel.

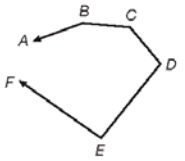


17. (Inv 1) If two parallel lines are cut by a transversal and one pair of same-side interior angles has angles that measure $(10x + 90)°$ and $(4x + 6)°$, what is the measure of each angle?

18. (12) **Write** In this figure, the transversal line ℓ intersects lines m and n. Write a paragraph explaining how you know that m and n are parallel.



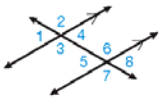
19. (15) This figure shows a polygon with one vertex and two sides missing.



a. Copy the figure and add a point G that makes $ABCDEFG$ concave.

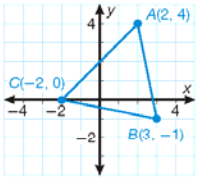
b. Make a second copy of the figure and add a point H that makes $ABCDEFH$ convex.

20. (Inv 1) Name every pair of corresponding angles in the diagram.



21. (8) **Building** A builder wants to add a diagonal beam to add support to a structure. The beam needs to extend across a height of 15 feet and a distance of 28 feet. What will the length of the beam be to the nearest hundredth of a foot?

22. (11) Determine the midpoint of each side of $\triangle ABC$.

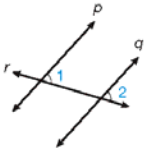


23. (13) **Algebra** The height of a triangle is 12.7 centimeters and its area is 31.75 square centimeters. Use the Triangle Area Formula to determine its base length.

24. (4) **Multiple Choice** Which statement is always true?

A Two planes intersect in a straight line.
B Two lines are contained in exactly one plane.
C Two lines can intersect at two points.
D Any four points can be contained in exactly one plane.

25. (12) **Multiple Choice** If $\angle 1$ and $\angle 2$ are congruent, which of these should be used to prove that lines p and q are parallel?



A Converse of the Alternate Exterior Angles Theorem
B Converse of the Alternate Interior Angles Theorem
C Converse of the Corresponding Angles Postulate
D Converse of the Same-Side Interior Angles Theorem

26. (7) **Predict** Use inductive reasoning to find the next term in this sequence. Explain the rule for the pattern.

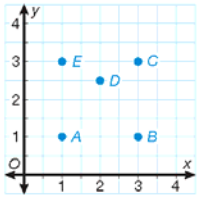
2, 3, 5, 9, 17, …

27. (8) **Multi-Step** Find the perimeter of a square if its area is 289 square centimeters.

28. (Inv. 1) If two parallel lines are intersected by a transversal, what is the sum of the measures of all four interior angles that are formed?

29. (8) Find the perimeter of a regular hexagon with side lengths of 6.8 inches

30. (15) If the points A, B, C, D, and E are connected, is polygon $ABCDE$ convex or concave?



LESSON 54

Representing Solids

Warm Up

1. (49) **Vocabulary** A prism with six square faces is called a _____.
2. (49) Name each of the pictured solids. If the solid is a prism or pyramid, classify it.

3. (49) According to Euler's Formula, if a polyhedron has 7 faces and 10 vertices, how many edges does it have?

New Concepts

In a **perspective drawing**, nonvertical parallel lines appear to meet at a point called a vanishing point. If you look straight down a highway, it appears that the edges of the highway eventually come together at a vanishing point, like point A in the diagram. In a perspective drawing, the **horizon** is the horizontal line that contains the vanishing point(s). A drawing with just one vanishing point is called **one-point perspective**.

Math Language

The **vanishing point** is the point in a perspective drawing on the horizon where parallel lines appear to meet.

Example 1 Drawing in One-Point Perspective

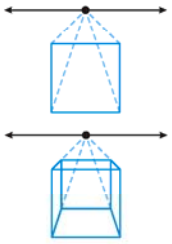
Draw a rectangular prism in one-point perspective. Use a pencil with an eraser.

SOLUTION

Step 1 Draw a square and a horizontal line above it representing the horizon. Mark a vanishing point on the horizon.

Step 2 Draw a dashed line from the vanishing point to each of the four corners of the square.

Step 3 Using the dashed lines drawn in Step 2, draw the sides of a smaller square.



Step 4 Connect the two squares and erase the reference lines and the horizon that are located behind the prism. This prism is drawn from a one-point perspective.

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A drawing with two vanishing points is said to have two-point perspective. Look at the following example to see how a drawing can be made from a two-point perspective.

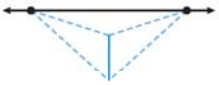
Example 2 **Drawing in Two-Point Perspective**

Draw a rectangular prism in two-point perspective in which the vanishing points are above the prism.

SOLUTION

Step 1 Draw a horizontal line that represents the horizon. Place two vanishing points on the horizon. Draw a vertical line segment below the horizontal line and between the two vanishing points, representing the front edge of the prism.

Step 2 Draw dashed lines from each vanishing point to the top and bottom of the vertical line as shown.



Step 3 Draw vertical segments between the dashed lines from Step 2 as shown and draw segments to connect them to the first segment.

Step 4 Draw dashed perspective lines from the segments drawn in Step 3 to each of the vanishing points as shown.

Step 5 Draw a dashed vertical line between the two intersections of the perspective lines just drawn. Sketch the segments that make the top of the prism.

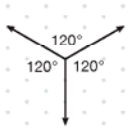
Step 6 Erase the horizon line and the dashed perspective lines. Keep the dashed lines inside the prism that represent the edges that are hidden.

This prism is drawn from a two-point perspective.

Math Reasoning

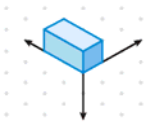
Model Could you also make a two-point perspective drawing by placing the vanishing points below the original line segment?

An **isometric drawing** is a way of drawing a three-dimensional figure using isometric dot paper, which has equally spaced dots in a repeating triangular pattern. The drawings can be made by using three axes that intersect to form 120° angles, as shown in the diagram.



Example 3 **Creating Isometric Drawings**

Create an isometric drawing of a rectangular prism.



SOLUTION

Draw the three axes on the isometric dot paper as shown above. Use this vertex as the bottom corner of the prism. Draw the box so that the edges of the prism run parallel to the three axes. Shading the top, front, and side of the prism will add the perception of depth.

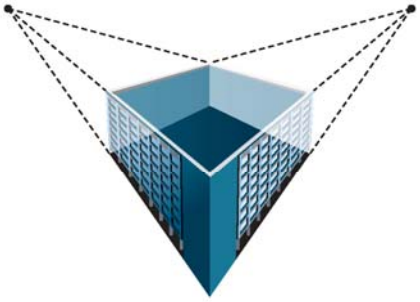
Example 4 **Application: Drafting**

Hint

In a two-point perspective drawing, it appears that one edge of the solid is the front of the diagram. In a one-point perspective drawing, it appears that a face of the solid is the front.

An architecture firm is planning to construct a rectangular building on a corner lot. The client would like a drawing that shows the building as though someone is looking at it from one edge. Should the drawing be from a one-point or two-point perspective? Make a sketch of what the drawing should look like.

SOLUTION



Since the front of the drawing will be an edge of the building, a two-point perspective drawing is appropriate. The diagram shows a completed view of the building.

Lesson Practice

(Ex 1) **a.** Draw a rectangular prism in one-point perspective in which the vanishing point is to the left of the square.

(Ex 2) **b.** Draw a cube in two-point perspective with the vanishing points and horizon below the vertical line.

(Ex 3) **c.** Make an isometric drawing of a triangular prism.

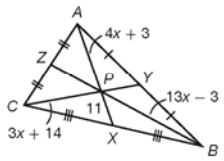
(Ex 4) **d.** Drafting Morgan wants to make a wooden bookshelf with two shelves. The bookshelf will be 1 meter wide, 1 meter deep, and 1.5 meters tall. To decide how much wood to buy, Morgan will draw his plans for the bookshelf. Should the drawing be from a one-point or two-point perspective? Sketch what Morgan's drawing should look like.

Practice Distributed and Integrated

* **1.** (54) Draw a triangular prism in one-point perspective so that the vanishing point is below the prism.

2. (52) **Write** Explain why the following statement is true.
If a quadrilateral is a square, then it is a rectangle.

3. (32) **Algebra** Find the length of $\overline{ZP}$ in the diagram.



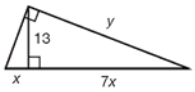
4. (42) What is the shortest distance from (5, 3) to the line $y = -2x + 8$?

* **5.** (54) **Architecture** An architect is creating different perspective drawings for a new building. The building is a rectangular prism and the client would like a drawing that focuses on the front façade of the building. Should the architect create the drawing using a one-point or two-point perspective? Sketch a sample drawing of the building.

6. (49) A figure has a hexagonal base and triangular lateral faces. Classify the figure.

7. (37) **Multi-Step** Graph the line and find the slope of the line that passes through the points $L(4, 1)$ and $M(3, -1)$. Then find a perpendicular line that passes through point $N(-2, -2)$.

8. (50) Find the value of x and y in the triangle shown to the nearest tenth.



9. (Inv 3) What is the sum of the exterior angles of a convex 134-sided polygon?

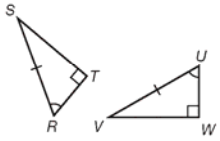
10. (52) Is the following statement always, sometimes, or never true?
A parallelogram is a rectangle.

***11.** (54) Trace the figure at right on your paper. Then locate the vanishing point and the horizon line.

12. (30) **Algebra** In $\triangle ABC$, $m\angle ABC = 90°$, $AB = (3x - 7)$, and $m\angle BCA = 60°$, and in $\triangle DEF$, $m\angle DEF = 90°$, $DE = (5x - 17)$, and $m\angle EFD = 60°$. What value of x will make $\triangle ABC \cong \triangle DEF$?

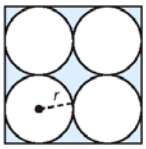
13. (32) The point where three or more lines intersect is the _____.

***14.** (36) Use the Hypotenuse-Angle Congruence Theorem to prove that $\triangle RST \cong \triangle UVW$.

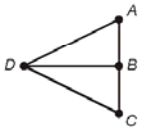


***15.** (53) Find the exact length of the hypotenuse of a 45°-45°-90° right triangle with a leg that is 57 feet long.

16. **Formulate** Four congruent circles are cut out of a square as shown. Write an expression for the area of the shaded region in terms of the radius of each circle, r. (40)



17. **Aviation** Four jet aircraft are flying in a triangular formation. Jets A, B, and C form a line perpendicular to the flight heading, while jet B is midway between the other two. Jet D flies directly in front of jet B. If $m\angle ADB = 37°$, what does the vertex angle of the triangular formation measure? Which theorem did you use? (51)



***18.** Use an indirect proof to prove that if two altitudes, $\overline{BX}$ and $\overline{CY}$ of $\triangle ABC$ are congruent, then the triangle must be isosceles. (48)

Given: $\overline{BX} \cong \overline{CY}$, $\overline{BX}$ and $\overline{CY}$ are altitudes.

Prove: $\triangle ABC$ isosceles.

***19.** Find the area, to the nearest hundredth, of a 45°-45°-90° right triangle with a hypotenuse of 17 centimeters. (53)

20. **Algebra** If a chord perpendicular to a radius cuts the radius in two pieces that are 7 and 2 inches long, respectively, what are the two possible lengths of the chord to the nearest tenth? (43)

***21.** **Justify** How does a two-point perspective differ when the vanishing points are located close together compared with when they are located further apart? Justify your reasoning with drawings. (54)

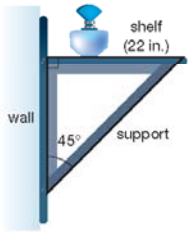
22. Find the geometric mean of $\sqrt{2}$ and 5. (50)

23. **Multiple Choice** If the diagonals of parallelogram $JKLM$ intersect at P, which of the following is true? (34)

A $JP = LP$ **B** $JP = KP$

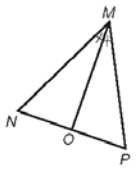
C $JL = KM$ **D** $JM = KM$

***24.** **Construction** The support of a shelf forms a 45°-45°-90° right triangle, with the shelf and the wall as the legs. Exactly how long is this support? (53)



***25.** **Analyze** $\triangle NPQ$ and $\triangle STV$ are similar isosceles triangles. How many of their six sides do you need numerical values for in order find all the other side lengths and the perimeters of both triangles? Explain. (44)

26. (38) Using the diagram on the right, find the length of $\overline{MP}$ if $OP = 5$, $NO = 8$, and $MN = 18$.

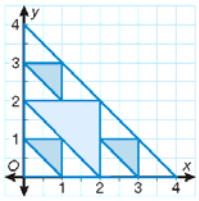


27. (Inv 4) **Cycling** Katya and Sareema start from the same location and bicycle in opposite directions for 2 miles each. Katya turns to her right 90° and continues for another mile. Sareema turns 45° to her left and continues for another mile. At this point, who is closer to the starting point?

28. (Inv 5) **Error Analysis** Darius drew this net of a number cube. Explain his error.

29. (45) **Analyze** Square $RSTU$ has vertices at $R(0, 4)$ and $S(0, 0)$. What are the possible coordinates of T and U?

30. (11) **Design** A white triangle with vertices at (0, 0), (4, 0), and (0, 4) is used to create a logo. A blue triangle is added to the design so that its vertices are the midpoints of the sides of the white triangle. The blue triangle divides the white triangle into three smaller white triangles. Smaller blue triangles are placed in each small white triangle so that their vertices are the midpoints of the sides of the small white triangles.



a. Find the coordinates of the vertices of the large blue triangle.

b. Find the coordinates of the vertices of each small blue triangle.

c. Which of the triangles are congruent, if any? Justify your answer.

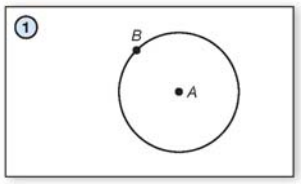
LAB
8

Tangents to a Circle

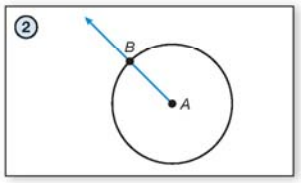
Construction Lab 8 *(Use with Lesson 58)*

Lesson 58 shows you how to identify lines tangent to a circle. This lab demonstrates how to construct lines tangent to a circle through a point on the circle.

1. To construct a tangent line through a given point on the circle, begin with $\odot A$ and a point on the circle, B.

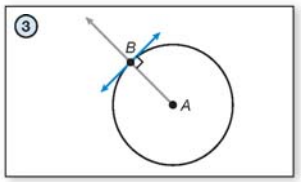


2. Draw $\overrightarrow{AB}$.



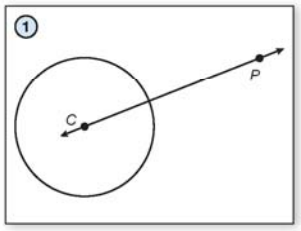
3. Using the method from Construction Lab 2, construct the line perpendicular to $\overrightarrow{AB}$ through B.

   The perpendicular line is tangent to $\odot A$ at B.



This lab demonstrates how to construct lines tangent to a circle through a point not on the circle.

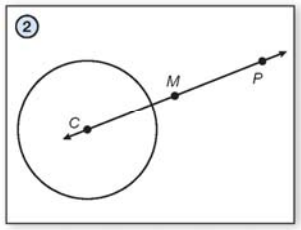
1. To construct two tangent lines through a point external to a circle, draw a circle and label the center C. Choose a point exterior to the circle, and label it P. Draw $\overleftrightarrow{CP}$.



Hint

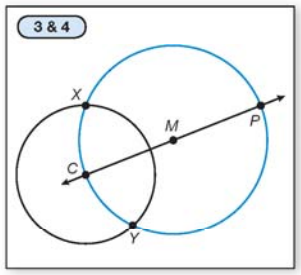
To construct the midpoint, use the method you learned in Construction Lab **3**.

(2.) Construct the midpoint of $\overline{CP}$. Label the midpoint M.

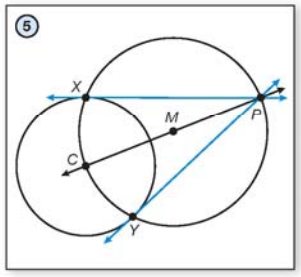


(3.) Draw a circle with a radius, CM centered on M. Notice that P is also on this circle.

(4.) Label the points of intersection of $\odot C$ and $\odot M$ as points X and Y.



(5.) Draw $\overleftrightarrow{XP}$. This line is tangent to $\odot C$ at point X. Draw $\overleftrightarrow{YP}$. Notice that $\overleftrightarrow{YP}$ is tangent to $\odot C$ at point Y.



Lab Practice

Use a compass to draw $\odot D$ and $\odot E$. Draw point A on $\odot D$. Draw points F and G outside, but near to, $\odot D$ and $\odot E$. Perform each construction indicated below.

a. a line tangent to $\odot D$ at point A

b. a line tangent to $\odot D$ from point F

c. a line tangent to $\odot E$ from point F

d. a line tangent to $\odot E$ from point G

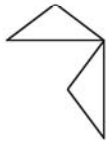
INVESTIGATION 8

Patterns

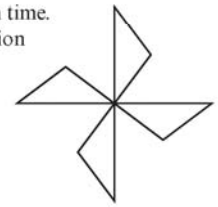
Finding patterns is a valuable problem-solving skill. In this investigation, you will study patterns made by transforming a figure. The basic figure we will work with is an isosceles triangle like the one shown here.

Copy this triangle by connecting the points (0, 0), (8, 0), and (4, 3) on a coordinate plane, and cut the triangle out so you can trace it onto paper.

Trace the cutout triangle onto a blank sheet of paper. First, transform the triangle by rotating it. With the triangle oriented as shown in the diagram above, rotate it 90° counterclockwise around either one of its acute angles. Trace the triangle again in its new position. You should now have a design like the one shown.



Continue to rotate the triangle 90° and trace it each time. Since the triangle will return to its original orientation after four rotations, the final design will look like the one shown.



1. What is the order of the rotational symmetry in the final pattern?
2. Does the final pattern have any lines of symmetry? If so, how many?

Trace your cutout triangle onto a blank sheet of paper. This time you will make a pattern by reflecting the triangle. Draw x- and y-axes and orient the triangle so its base lies on the x-axis and one of its acute angles lies on the origin.

3. Reflect the triangle over the y-axis and draw the resulting pattern.
4. Reflect the pattern from step 3 over the x-axis and draw the resulting pattern.
5. Does the final pattern have rotational symmetry? If so, what is the order of rotational symmetry?
6. Does the final pattern have any lines of symmetry? How many?

Translation symmetry is a type of symmetry describing a figure that can be translated along a vector so that the image coincides with the preimage. A **frieze pattern** is a pattern that has translation symmetry along a line.

Trace your cutout triangle onto a blank sheet of paper. Orient the triangle so its base is parallel to the bottom edge of the paper. Translate the triangle to the right until the vertices of the opposite acute angles lie on the same point, as in the figure shown.

Math Reasoning

Model Could you have used different transformations of the triangle to obtain the same image as the one in step 4? Explain how.

Translate the triangle to the right again. Continue this process until you have 4 triangles in a row. This is a frieze pattern.

7. What other transformation(s) could have been used to create this same pattern?

8. Does the final pattern have rotational symmetry? If so, what is the order of rotational symmetry?

9. Does the final pattern have any lines of symmetry? How many?

Now we will explore some geometric patterns. Draw two points. There is only one segment that can be drawn connecting these two points. What about 3 points? Draw 3 noncollinear points and draw segments connecting them. You find that there are three segments that can be drawn.

10. Draw four non-collinear points. Draw a line segment between each pair of points. How many line segments do you have?

11. **Predict** Based on the pattern you have seen so far, predict how many line segments you can draw connecting 5 noncollinear points. What about 6 noncollinear points? … 7 noncollinear points?

12. **Formulate** Write a rule for the number of line segments, L, between n points, in terms of the number of line segments between $n - 1$ points (denoted L_{n-1}).

The numbers in the series you have just discovered are called triangular numbers. Triangular numbers are numbers that are equal to the sum of the first n whole numbers. The first 8 triangular numbers are: 1, 3, 6, 10, 15, 21, 28, and 36.

Can an algebraic expression be written for the n^{th} triangular number? The n^{th} triangular number is given by the formula below.

$x = 1 + 2 + 3 + 4 + \ldots + n - 1 + n$

To find an expression for the n^{th} triangular number, take this series and add it to itself. Instead of adding the terms in order though, add the first term to the last term, the second term to the second to last term, and so on.

$$\begin{array}{r} x = 1 + 2 + 3 + 4 \ldots + n - 1 + n \\ +\, x = 1 + 2 + 3 + 4 \ldots + n - 1 + n \\ \hline 2x = (1 + n) + (2 + n - 1) + (3 + n - 2) + \ldots \end{array}$$

Now notice that in the sum, the expressions in the parenthesis can be simplified. After being simplified, the sum becomes $2x = (n + 1) + (n + 1) + \ldots$

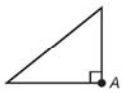
Each expression in parenthesis is the same. Moreover, we know that the series has n terms. So it can be simplified further, resulting in $2x = n(n + 1)$. To solve for x, which is the n^{th} triangular number, divide by 2.

$$x = \frac{n(n + 1)}{2}$$

13. Using the expression above, what is the 50^{th} triangular number? What is the 100^{th} triangular number?

Investigation Practice

a. Using the right triangle given here, sketch the result of rotating the figure 90° counterclockwise about the point A.



b. Continue to rotate the triangle 90° until it coincides with itself, sketching the result of each rotation. What is the order of rotational symmetry in the final figure? Does it have any lines of symmetry?

c. Return to the initial figure and reflect it over the vertical leg of the triangle. Then reflect it over the horizontal leg of the triangle. What kind of polygon is the resulting figure?

d. Does the resulting figure have any lines of symmetry? Does it have rotational symmetry?

e. Square numbers are whole numbers that could be the area of a square. The series begins: 1, 4, 9, 16, 25, …. Write an equation to find the n^{th} square number.

f. What is the 30th square number?