LESSON 58

Completing the Square

Warm Up

1. (11) **Vocabulary** In the trinomial $ax^2 + bx + c$, c is the _____ term.
2. (23) Factor the expression $x^2 + 2x + 1$.
3. (5) True or False. If a number x is a perfect square, then $\sqrt{x}$ is a whole number.

New Concepts

Recall that a perfect square trinomial can be factored into a squared binomial.

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

The Square Root Property can be used in solving quadratic equations in the form of a binomial square.

Square Root Property
If $x^2 = a$, where $a > 0$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$. In general, if $x^2 = a$, then $x = \pm\sqrt{a}$ for any $a > 0$.

Example 1 **Solving Quadratic Equations That Are Perfect Squares**

a. Solve $x^2 - 10x + 25 = 9$.

SOLUTION

Step 1: Factor the perfect square trinomial.

$$x^2 - 10x + 25 = 9$$
$$x^2 - 2(5)x + 5^2 = 9$$
$$(x - 5)^2 = 9$$

Step 2: Apply the Square Root Property.

$$(x - 5)^2 = 9$$
$$x - 5 = \pm\sqrt{9}$$
$$x - 5 = \pm 3$$

Step 3: Solve for x.

$$x - 5 = 3 \qquad x - 5 = -3$$
$$+5 \quad +5 \qquad +5 \quad +5$$
$$x = 8 \qquad x = 2$$
$$x = 8 \quad \text{or} \quad x = 2$$

Caution

Do not forget that the Square Root Property yields exactly **two** solutions.

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b. Solve $x^2 + 24x + 144 = 5$.

SOLUTION

Step 1: Factor the perfect square trinomial.

$$x^2 + 24x + 144 = 5$$
$$x^2 + 2(12)x + 12^2 = 5$$
$$(x + 12)^2 = 5$$

Step 2: Apply the Square Root Property.

$$(x + 12)^2 = 5$$
$$x + 12 = \pm\sqrt{5}$$

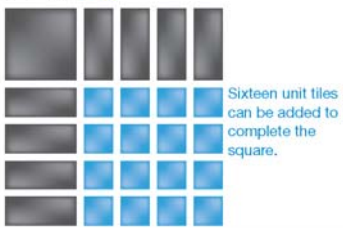
Since 5 is not a perfect square, it can be left as $\sqrt{5}$.

Step 3: Solve for x.

$$x + 12 = \pm\sqrt{5}$$
$$-12 \quad -12$$
$$x = -12 \pm \sqrt{5}$$
$$x = -12 + \sqrt{5} \quad \text{or} \quad x = -12 - \sqrt{5}$$

Some quadratic equations cannot be solved by factoring. In these cases the method of completing the square can be used. To **complete the square,** a term can be added to a quadratic expression of the form $x^2 + bx$ to form a perfect square trinomial.

Exploration **Modeling Completing the Square**

The model represents the quadratic binomial $x^2 + 8x$. $x^2 + 8x$ 	How many more unit tiles should be added to the model to create a square? $x^2 + 8x + 16$ 
The result is a perfect square trinomial.	$x^2 + 8x + 16 = (x + 4)^2$

Draw a model of $x^2 + 10x$. Use your model to find a perfect square trinomial and its factors.

Completing the Square

Given a quadratic of the form $x^2 + bx$, add to it the square of half the coefficient of x, $\left(\frac{b}{2}\right)^2$, to create a perfect square trinomial.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 2 Completing the Square

Complete the square and factor the resulting perfect square trinomial $x^2 + 18x$.

SOLUTION Add the square of half the coefficient of x to the binomial. The coefficient of x is 18. Add $\left(\frac{18}{2}\right)^2 = 9^2$ to the binomial to create a perfect square trinomial.

$$x^2 + 18x + 9^2 = x^2 + 18x + 81 = (x + 9)^2$$

When using the method of completing the square to solve a quadratic equation, keep in mind that when you add a constant term to one side of an equation you must also add that constant to the other side of the equation.

Example 3 Solving Quadratic Equations by Completing the Square

Solve $x^2 - 14x - 8 = 0$ by completing the square.

SOLUTION

Step 1: $x^2 - 14x - 8 + 8 = 0 + 8$
$x^2 - 14x = 8$ Isolate the binomial $x^2 - 14x$.

Step 2: Add the square of half the coefficient of x to both sides of the equation.

The coefficient of x is -14; $\left(\frac{-14}{2}\right)^2 = (-7)^2 = 49$.

$x^2 - 14x + 49 = 8 + 49$
$x^2 - 14x + 49 = 57$

Step 3: $x^2 - 14x + 49 = 57$
$x^2 - 2(7)x + 7^2 = 57$ Factor the perfect square trinomial.
$(x - 7)^2 = 57$

Step 4: $x - 7 = \pm\sqrt{57}$ Apply the Square Root Property.

Step 5: $x - 7 + 7 = \pm\sqrt{57} + 7$
$x = 7 \pm \sqrt{57}$ Solve for x.

Math Reasoning

Estimate Without using a calculator, estimate the decimal approximation for the solutions to Example 3.

Example 4 **Application: Projectile Motion**

During a baseball game a player hits a fly ball. The height $f(t)$, in feet, of the ball at time t, in seconds, can be described by the function $f(t) = -16t^2 + 64t + 4$. Write the equation in vertex form by completing the square. Then determine the maximum height of the ball and the time it takes to reach the maximum height.

SOLUTION

Complete the square for the trinomial expression that describes the height of the ball.

$f(t) = -16t^2 + 64t + 4$

$= (-16t^2 + 64t) + 4$ Group the binomial $-16t^2 + 64t$

$= -16(t^2 - 4t) + 4$ Factor out -16 from $(-16t^2 + 6t)$.

$= -16(t^2 - 4t + 4) + 4 + \boxed{64}$ Complete the square.

$= -16(t - 2)(t - 2) + 68$ Factor. Then simplify.

$f(t) = -16(t - 2)^2 + 68$ Write the function in vertex form.

To determine the maximum height of the ball use the vertex form of a function. Recall the vertex form $f(x) = a(x - h)^2 + k$ of a quadratic function. The point (h, k) is the vertex.

For the function $f(t) = -16(t - (+2)^2 + 68$, the vertex is $(2, 68)$. Therefore, the ball reaches a maximum height of 68 feet 2 seconds after the ball was hit.

Check Use a table to graph the function. Then estimate the vertex of the parabola on the graph.

x	0	1	2	3	4
$f(x)$	4	52	68	52	4

The graph verifies that the vertex occurs at (2, 68)

Caution

When completing the square, be sure to keep the equation equivalent to the original equation. Notice $(-16)(4)$ was added to complete the square, so $+64$ was added to keep the equation equivalent to the original equation.

Lesson Practice

(Ex 1) **a.** Solve $x^2 - 2x + 1 = 36$.

(Ex 1) **b.** Solve $x^2 + 16x + 64 = 6$.

(Ex 2) **c.** Complete the square and factor the resulting perfect square trinomial

$$x^2 - 20x.$$

(Ex 3) **d.** Solve $x^2 + 26x - 2 = 0$ by completing the square.

(Ex 4) **e.** A pebble is thrown into the air from ground level with an initial velocity of 20 feet per second. The height of the pebble h at any given time t can be described by the equation

$$h = -16t^2 + 20t.$$

Find the time at which the height of the pebble is 4 feet. Round your answer to the hundredths place.

Practice Distributed and Integrated

Solve by using either substitution or elimination.

1. (24) $2y - 2x = 8$, $y + x = -2$

2. (24) $y - 2x = 1$, $y = -2$

3. (24) $3x + 2y = 12$, $5x - 4y = 8$

4. (10) Solve the inequality $-3 < d + 2$ and $d + 2 < 12$.

5. (10) Solve and graph the inequality $-4 \leq 4x - 8 < 12$.

Determine the kind of variation, if any, for each equation.

6. (12) $y = \frac{1}{x}$, for $x \neq 0$

7. (12) $z = \frac{1}{xy + a}$, for $x, y \neq 0$ and any value of a

Solve.

***8.** (58) $x^2 - 8x + 16 = 3$

***9.** (58) $x^2 + 26x + 169 = 81$

10. (30) **Physics** An object dropped from a height 4 meters above the ground after x seconds is modeled by the function $y = -4.9x^2 + 4$. If measured in feet, the height is modeled by $y = -16x^2 + 1.2$. Without graphing, tell how the graphs of each function compare to each other.

11. (22) **Error Analysis** Sam listed the following points $\{(2, 3), (1, 6), (5, -3), (1, 2)\}$ when giving an example of a discrete function. Correct his example.

12. (51) **Multiple Choice** What is the result of dividing $x^3 - 7x + 3$ by $x - 2$?

A $x^2 + 2x + 3 - \frac{3}{x - 2}$

B $x^2 + 2x - 3$

C $x^2 + 2x - 3 - \frac{3}{x - 2}$

D $x^2 + 2x$

***13.** (58) **Multi-Step** Use $x^2 + 104x - 97 = 4$.

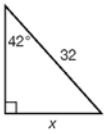
a. What is the first step before you can complete the square?

b. What constant can be added to the binomial to create a perfect square trinomial?

c. What is the factored form of the perfect square trinomial?

d. Solve the equation for x.

14. (46) **Justify** Show that it is possible to solve for x by using either the sine or cosine function.



*15. (57) **Formulate** Formulate an expression for the amount Stu will have in his bank account at the end of a year if he invests a principal of $1,850 at an annual interest rate of 2.4%, compounded monthly.

*16. (58) **Multiple Choice** Chose the letter that best represents the value of x in the equation $x^2 - 42x + 441 = 2$.

A $x = -21 \pm \sqrt{2}$ **B** $x = 20, 22$

C $x = 21 \pm \sqrt{2}$ **D** $x = -20, -22$

17. (26) **Geometry** Miriam is going to create a rectangular vegetable garden along the side of her house and is going to enclose it within a fence on three sides. If the total amount of fencing she purchases is 80 feet, write an equation in standard form for finding all possible lengths and widths for the garden.

18. (19) **Write** Explain how to multiply $(2x + 8)(3x - 6)$ using the FOIL method.

19. (45) **Formulate** Describe the correlation coefficient r using a compound inequality and absolute value symbols.

*20. (55) **Pass Codes** A computer generates temporary pass codes to users. Each pass code is exactly 3 letters long and it can have repeating letters. What is the probability of getting a pass code with 3 vowels (A, E, I, O, or U)?

21. (23) **Deck Building** The area of a triangular section of a deck is $72x^2 - 98$ ft^2. If the base of the section of deck is $12x + 14$ ft, find the height of the section.

22. (52) **Agriculture** Lee and his brother own a piece of land that has the shape of an equilateral triangle with side length 200 m. They divide the land between them into two equal parts along the altitude of the triangle by putting a fence. What is the length of the fence? What area of land does each of them own? Use a graphing calculator and round your answers to the nearest hundredth.

*23. (54) **Graphing Calculator** Describe the feasible region for the set of inequalities

$$x \geq 0,\ y \geq 0,\ y \leq -0.75x + 8.$$

24. (24) **Error Analysis** Find and explain the error in solving the system $\begin{aligned} x - 10y &= 10 \\ 2x - 20y &= 23 \end{aligned}$ in the step below.

$$\begin{array}{rr} -2(x - 10y = 10) & -2x + 20y = -20 \\ \underline{+2x - 20y = 23} & \underline{+2x - 20y = 23} \\ & 40y = 3 \end{array}$$

25. (50) **Pendulums** The period T (in seconds) of a simple pendulum as a function of its length l (in meters) is given by the equation $T = 2\pi\sqrt{\frac{l}{9.8}}$. Express the length l as a function of the time t.

*26. (57) **Multi-Step** The half-life of Uranium-234 is 245,000 years.

a. Formulate an exponential function representing the amount of a grams Uranium-234 remaining after t years.

b. Use the function to find how much of a 150-gram sample of Uranium-234 will be left after 182,000 years.

27. (53) Let $f(x) = 3x$ and $g(x) = 3x + 2$. Find the composite function $g(f(x))$.

28. (40) **Justify** A student saw his friend write that $-\sqrt{81} = -9$, and he told his friend that it is wrong because a square root cannot yield a negative number. Why is the student wrong and his friend correct?

29. (44) **Generalize** How might you rationalize the denominator for the expression $\frac{1}{\sqrt[3]{5}}$, when there is a *cube* root in the denominator?

*30. (58) **Projectile Motion** The world record for the World Championship Punkin Chunkin, held each year in the state of Delaware, is held by an air cannon named "2nd Amendment." The height of a pumpkin t seconds after being shot from this machine can be described by the equation $h = -16t^2 + 880t + 70.7$. After how many seconds will a pumpkin shot from this machine land on the ground? Round your answer to the tenths place.

LESSON 69

Simplifying Complex Expressions

Warm Up

1. (62) **Vocabulary** A number that can be written in the form $a + bi$ where a and b are real numbers is a _____.

2. (19) Multiply $(3x - 5)(4x + 2)$.

3. (40) Simplify $\frac{5}{\sqrt{5}}$.

4. (62) Add $(3 + 2i) + (6 - 8i)$.

New Concepts

Complex numbers $(a + bi)$ can be graphed on a complex plane, where the horizontal axis represents the real part, a, and the vertical axis represents the imaginary part, bi.

Math Language

The **imaginary axis** is sometimes referred to as the ***i*-axis.**

Example 1 Graphing Complex Numbers

Graph each number on the complex plane.

a. $3 + 4i$ b. $-2 - 3i$

c. $-4i$ d. 1

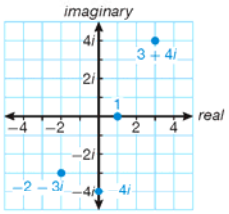
SOLUTION

a. Move 3 units to the right and 4 units up.

b. Move 2 units to the left and 3 units down.

c. Move 0 units right or left and 4 units down.

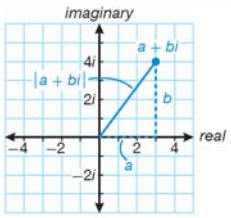
d. Move 1 unit to the right and 0 units up or down.



The **absolute value of a complex number** is its distance from the origin. Unless the point is on the x- or i-axis, the distance is the length of a line segment that forms the hypotenuse of a right triangle and can be found using the Pythagorean Theorem.

Absolute Value of a Complex Number

$$|a + bi| = \sqrt{a^2 + b^2}$$



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Example 2 **Finding the Absolute Value of Complex Numbers**

Find the absolute value of each number.

a. $3 + 4i$

SOLUTION $|a + bi| = \sqrt{a^2 + b^2}$

$$|3 + 4i| = \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16} = \sqrt{25} = 5$$

b. $-2 - 3i$

SOLUTION $|a + bi| = \sqrt{a^2 + b^2}$

$$|-2 - 3i| = \sqrt{(-2)^2 + (-3)^2}$$
$$= \sqrt{4 + 9} = \sqrt{13}$$

c. $-5i$

SOLUTION $|a + bi| = \sqrt{a^2 + b^2}$

$$|-5i| = \sqrt{0^2 + (-5)^2}$$
$$= \sqrt{25} = 5$$

Math Reasoning

Verify Use the formula for the absolute value of a complex number to show that $|-1|$ is 1.

When multiplying complex numbers, remember that $i^2 = -1$.

Example 3 **Multiplying Complex Numbers**

Multiply. Write answers in the form $a + bi$.

a. $-4i(7 + 3i)$

SOLUTION

$-4i(7 + 3i)$	
$-28i - 12i^2$	Distributive Property
$-28i - 12(-1)$	$i^2 = -1$
$-28i + 12$	Simplify.
$12 - 28i$	Write in standard form.

b. $(5 + 4i)(6 - 2i)$

SOLUTION

$(5 + 4i)(6 - 2i)$	
$30 - 10i + 24i - 8i^2$	Use the FOIL method.
$30 + 14i - 8i^2$	Simplify.
$30 + 14i - 8(-1)$	$i^2 = -1$
$30 + 14i + 8$	Simplify.
$38 + 14i$	Write in standard form.

Study the powers of i below.

Powers of i	
$i^1 = i$	$i^5 = i \cdot i^4 = i(1) = i$
$i^2 = -1$	$i^6 = i \cdot i^5 = i(i) = i^2 = -1$
$i^3 = i \cdot i^2 = i(-1) = -i$	$i^7 = i \cdot i^6 = i(-1) = -i$
$i^4 = i \cdot i^3 = i(-i) = -i^2 = -(-1) = 1$	$i^8 = i \cdot i^7 = i(-i) = -i^2 = -(-1) = 1$

Notice that the four simplified values of i, -1, $-i$, and 1 repeat such that $i = i^5$, $i^2 = i^6$, $i^3 = i^7$, and $i^4 = i^8$. Any power of i can be found by dividing the exponent of i by four and using the remainder.

When the remainder is …	The expression is equivalent to …
0	$i^4 = 1$
1	$i^1 = i$
2	$i^2 = -1$
3	$i^3 = -i$

Example 4 **Evaluating Powers of *i***

Simplify each expression.

a. i^{30}

SOLUTION $30 \div 4 = 7$ R 2, so $i^{30} = i^2 = -1$.

b. $5i^{12}$

SOLUTION $12 \div 4 = 3$ R 0, so $i^{12} = i^4 = 1$. $5i^{12} = 5(1) = 5$

c. $-2i^{13}$

SOLUTION $13 \div 4 = 3$ R 1, so $i^{13} = i^1 = i$.

$-2i^{13} = -2(i) = -2i$

Caution

A simplified expression does not have a radical in the denominator. Because $i = \sqrt{-1}$, an expression with i in the denominator is not simplified.

To eliminate a radical in a denominator, multiply both the numerator and denominator of the fraction by the conjugate of the denominator. The **complex conjugate** of $a + bi$ is $a - bi$.

Example 5 **Dividing Complex Numbers**

a. Divide 3 by $6 + i\sqrt{5}$. Write the answer in the form $a + bi$.

SOLUTION

$$\frac{3}{6 + i\sqrt{5}}$$

$$\frac{3}{6 + i\sqrt{5}}\left(\frac{6 - i\sqrt{5}}{6 - i\sqrt{5}}\right)$$

Multiply the numerator and the denominator by the complex conjugate of the denominator.

$$\frac{18 - 3i\sqrt{5}}{36 - 5i^2}$$

Distribute and simplify.

$$\frac{18 - 3i\sqrt{5}}{41} = \frac{18}{41} - \frac{3\sqrt{5}}{41}i$$

b. Divide $4 + i$ by $6i$. Write the answer in the form $a + bi$.

SOLUTION $\frac{4+i}{6i}$

$\frac{4+i}{6i}\left(\frac{-6i}{-6i}\right)$	Multiply the numerator and the denominator by the complex conjugate of the denominator.
$\frac{-24i-6i^2}{-36i^2}$	Distribute.
$\frac{-24i-6(-1)}{-36(-1)}$	Simplify.

$$\frac{-24i+6}{36} = -\frac{24}{36}i + \frac{6}{36} = -\frac{2}{3}i + \frac{1}{6} = \frac{1}{6} - \frac{2}{3}i$$

Math Reasoning

Analyze Why is $-6i$ the conjugate of $6i$?

Lesson Practice

Graph each number on the complex plane. (Ex 1)

a. $1 - 3i$ **b.** $-2 + 4i$ **c.** -3 **d.** $2i$

Find the absolute value of each number. (Ex 2)

e. $7 - 2i$ **f.** $-5 + i$ **g.** $10i$

Multiply. Write answers in the form $a + bi$. (Ex 3)

h. $8i(-9 - 5i)$ **i.** $(2 - 3i)(5 + 9i)$

Simplify each expression. (Ex 4)

j. i^{18} **k.** $-3i^{11}$ **l.** $\frac{1}{2}i^{21}$

m. (Ex 5) Divide $3 + 3i$ by $3i$. Write the answer in the form $a + bi$.

n. (Ex 5) Divide -4 by $2 + i\sqrt{3}$. Write the answer in the form $a + bi$.

Practice Distributed and Integrated

Solve using the quadratic formula.

1. (65) $f(x) = x^2 + 8x - 3$ **2.** (65) $f(x) = 3x^2 - 10x + 4$

3. (55) **Probability** A bag holds 53 yellow marbles, 17 red marbles, and 30 green marbles. What is the probability of not choosing a red marble?

4. (57) A savings account earns interest at an annual rate of 3.8%, compounded quarterly. If the account begins with a principal amount of $2,700, what will its value be after 14 years?

Solve for x.

5. (59) $x^{\frac{1}{7}} = 5$ **6.** (59) $x^{\frac{2}{3}} = 36$

*7. (69) **Multiple Choice** In which quadrant of the complex plane is $5 - i$ located?
A Quadrant I **B** Quadrant II **C** Quadrant III **D** Quadrant IV

8. (61) **Write** Explain how to use synthetic division to determine if $x + 6$ is a factor of $x^3 + 3x^2 - 28x - 60$ and then perform the division.

Simplify.

9. (62) $\frac{1}{4}\sqrt{-256}$.

10. (62) $\frac{1}{2}\sqrt{-2500}$.

11. (56) Given $\theta = -920°$, find the measure of the reference angle.

*12. (69) **Generalize** Show that the product of any two complex conjugates, $a + bi$ and $a - bi$, is a real number.

*13. (67) Evaluate the inverse trigonometric function $\text{Sin}^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. Give your answer in both radians and degrees.

14. (Inv 4) Why can't the matrix below be used as an encoding matrix for a cipher?

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 13 \end{bmatrix}$$

15. (56) **Amusement Parks** You get on a ferris wheel ride, and the ferris wheel turns through 1370°. How much further must the wheel turn before you are at the bottom again?

*16. (68) **Error Analysis** Find and correct the error a student made in finding $P(\text{AM}|\text{Male})$.

$$P(\text{AM}|\text{Male}) = \frac{16}{20} = \frac{4}{5} = 80\%$$

	Male	Female
AM	16	4
PM	8	12

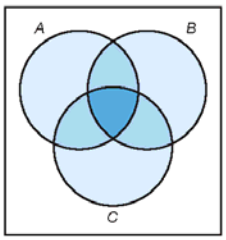
*17. (66) **Graphing Calculator** **a.** Graph $y = x^3 + 6x^2 - 32$ on a graphing calculator. What are the x-intercepts?

b. Find the multiplicity of each root of $x^3 + 6x^2 - 32 = 0$ by factoring.

*18. (68) **Formulate** Give a rule for finding $P(B|A)$. *Hint:* Consider the rule for $P(A \text{ and } B)$ for dependent events.

19. (64) **Multiple Choice** Which equation is equivalent to $10^x = 0.1$?
A $\log_{0.1} 10 = x$ **B** $\log_{0.1} x = 10$
C $\log_{10} 0.1 = x$ **D** $\log_{10} x = 0.1$

20. (60) **Formulate** Write a formula for $P(A \text{ or } B \text{ or } C)$. Use the diagram to the right to help you.



21. (67) **Multi-Step** Let $f(\theta) = \text{Sin}\theta$ and $g(\theta) = \text{Sin}^{-1}\theta$, where θ is in radians.

a. Evaluate $f\left(g\left(\frac{1}{2}\right)\right)$. Show your steps.

b. Evaluate $g\left(f\left(\frac{\pi}{6}\right)\right)$. Show your steps.

c. Based on part a, what is the value of $f(g(x))$? Based on part b, what is the value of $g(f(x))$?

22. (58) **Error Analysis** Examine the work below. Find and explain the error. Then find the correct solution.

$$x^2 - 44x + 484 = 121$$
$$(x - 22)^2 = 121$$
$$x - 22 = 11$$
$$x = 33$$

***23.** (65) **Probability** A student graphs $y = 9x^2 - 29x + 22$ and uses a graphing calculator to find the value of the zeros one at a time. What is the probability that the first zero the student finds is negative? Why?

***24.** (69) **Multi-Step** Simplify $i^{22} + i^{15}$. Then graph the number on the complex plane.

25. (63) **Planet Rotation** Venus is only slightly smaller than Earth; its radius is approximately 3757 miles. But it rotates much more slowly; it takes 243 Earth days for Venus to make one complete rotation. If an object is fixed on Venus's equator, how far does it travel in 1 hour due to Venus's rotation?

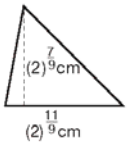
26. (64) **Analyze** Describe the value(s) of b^x when $b = 1$ and when $b \neq 1$.

***27.** (69) Find the absolute value of $|3 - 4i|$.

***28.** (69) Find the absolute value of $|6 + 2i|$.

29. (62) **Sports** A football player playing in a domed stadium kicks a football straight up into the air. The height of the football after t seconds can be modeled by $-16t^2 + 80t$. The dome has a height of about 288 feet. The solutions of $-16t^2 + 80t = 288$ give the number of seconds it will take for the football to reach the top of the dome. Solve the equation. Will the football reach the top of the dome? If so, when?

30. (59) **Geometry** Find the area of the triangle to the right.



LESSON 95

Factoring Higher-Order Polynomials

Warm Up

(35) 1. **Vocabulary** If $f(x) = ax^2 + bx + c$ is divisible by $(x - d)$, then $(x - d)$ is a _____ of $f(x)$.

(51) 2. Use synthetic division to test if $6x^3 - 5x^2 - 34x + 40$ is divisible by $(x - 2)$.

(51) 3. Use synthetic division to test if $20x^3 - 53x^2 - 122x + 56$ is divisible by $(x - 2)$.

New Concepts

A constant a is a root of polynomial $P(x)$ if $P(a) = 0$.

You can use the Reminder Theorem and the Factor Theorem to test for roots of a polynomial.

Remainder Theorem

The **Remainder Theorem** states:

If the polynomial function $P(x)$ is divided by $x - a$, then the remainder r is $P(a)$.

Math Reasoning

Generalize Suppose a polynomial $P(x)$ has no real roots. What does the Factor Theorem state about such a polynomial?

Factor Theorem

The **Factor Theorem** states:

For any polynomial $P(x)$,$(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Use the Factor Theorem to test if a value is a root of a polynomial.

Example 1 Using the Factor Theorem to Test for Roots

a. Determine if $x = 5$ is a root of $P(x)$.

$$P(x) = 2x^6 - 43x^5 + 75x^4 + 1765x^3 - 857x^2 - 22{,}542x - 30{,}240$$

SOLUTION

Use synthetic division to test if $r(5) = 0$. Divide $P(x)$ by $(x - 5)$.

5	2	−43	75	1765	−857	−22,542	−30,240
		10	−165	−450	6,575	28,590	30,240
	2	−33	−90	1,315	5,718	6,048	0

Since $P(5) = r(5) = 0$, 5 is a root of the polynomial.

Online Connection
www.SaxonMathResources.com

b. Determine if $x = -7$ is a root of $P(x)$.

$$P(x) = 9x^8 - 101x^7 + 195x^6 + 622x^5 - 621x^4 - 1565x^3 - 1947x^2 + 2772x + 2940$$

SOLUTION

Use synthetic division to test if $r(-7) = 0$. Divide $P(x)$ by $(x + 7)$.

−7	9	−101	195	622	−621	−1565	−1947	2772	2940
		−63	1148	−9401	61,453	−425,824	2,991,723	−20,928,432	146,479,620
	9	−164	1343	−8779	60,832	−427,389	2,989,776	−20,925,660	146,482,560

Since $P(-7) = r(-7) \neq 0$, -7 is not a root of the polynomial.

Example 2 Finding Roots of $P(x)$

Find all the rational roots of $P(x) = x^5 + 4x^4 - 10x^2 - x + 6$.

SOLUTION

By the Rational Root Theorem, the possible rational roots are ± 1, ± 2, ± 3, ± 6.

Use synthetic division to test the roots.

1	1	4	0	−10	−1	6
		1	5	5	−5	−6
	1	5	5	−5	−6	0

Since $P(1) = 0$, 1 is a root of the polynomial.

$P(x) = (x - 1)(x^4 + 5x^3 + 5x^2 - 5x - 6)$.

By the Rational Root Theorem, the possible rational roots of the quartic quotient are ± 1, ± 2, ± 3, ± 6. Use synthetic division to test the roots.

1	1	5	5	−5	−6
		1	6	11	6
	1	6	11	6	0

Since $P(1) = 0$, 1 is a root of the polynomial with multiplicity 2.

$P(x) = (x - 1)(x - 1)(x^3 + 6x^2 + 11x + 6)$.

By the Rational Root Theorem, the possible rational roots of the cubic quotient are ± 1, ± 2, ± 3, ± 6. Use synthetic division to test the roots.

1	1	6	11	6
		1	7	18
	1	7	18	24

−1	1	6	11	6
		−1	−5	−6
	1	5	6	0

Since $P(-1) = 0$, -1 is a root of the polynomial.

$P(x) = (x - 1)(x - 1)(x + 1)(x^2 + 5x + 6)$

The last quotient is a quadratic that can be factored.

$P(x) = (x - 1)(x - 1)(x + 1)(x + 2)(x + 3)$

The roots are **1**, **−1**, **−2**, and **−3**. The root 1 has a multiplicity 2.

Math Reasoning

Analyze Suppose that a polynomial $P(x)$ has two real roots a and b. Could $P(x)$ be of degree 3?

Example 3 **Application: Braking Distance**

When a car needs to be brought to an immediate stop, it takes time for someone to step on the brake and time for the car to come to a complete stop. During these times, the car is still moving. The stopping distance is how far the car travels during this time frame. A reasonable mathematical model for the stopping distance d, in feet, based on the car's speed s, in miles per hour, is shown below. Use the Remainder Theorem to evaluate $d(20)$, $d(30)$, and $d(55)$.

$d(s) = 0.05s^2 + 2.2s$

SOLUTION Use synthetic division.

20	0.05	2.2	0
		1	64
	0.05	3.2	64
30	0.05	2.2	0
		1.5	111
	0.05	3.7	111
55	0.05	2.2	0
		2.75	272.25
	0.05	4.95	272.25

The stopping distances are $d(20) = 64$ ft, $d(30) = 111$ ft $d(55) = 272.25$ ft.

Lesson Practice

Determine if $x = 7$ is a root of $P(x)$.

(Ex 1) **a.** $P(x) = 2x^6 - 43x^5 + 75x^4 + 1765x^3 - 857x^2 - 22{,}542x - 30{,}240$

b. $P(x) = 9x^8 - 101x^7 + 195x^6 + 622x^5 - 621x^4 - 1565x^3 - 1947x^2 + 2772x + 2940$

(Ex 2) **c.** Find all the rational roots of $P(x) = x^3 + 4x^2 + 1x - 6$.

(Ex 3) **d.** The height of an object thrown at 40 feet per second can be modeled by $h(t) = -16t^2 + 40t$ where h is the height in feet and t is time in seconds. Use the remainder theorem to find the height of the object after 2 seconds.

Practice Distributed and Integrated

Solve.

1. (84) $x - \frac{6}{x} = 1$

2. (84) $\frac{x^2 + x - 6}{x + 1} = 0$

3. (84) $\frac{7x}{3x + 2} = 2$

4. (57) Does the graph of $y = b^x + k$, through the points $(3, -3)$ and $(5, 0)$, model exponential growth or decay?

***5.** (94) Solve $\frac{1}{x^3} \geq 5$. Round to the nearest thousandth.

6. (8) In a certain recipe, the amount of sugar is directly proportional to the amount of flour. If 3 cups of sugar are used with 8 cups of flour, how many cups of sugar are used with 12 cups of flour?

***7.** (95) **Error Analysis** Two students were evaluating $P(-4)$ for the polynomial $P(x)$, but they got different results. Which student made the mistake?

$$P(x) = x^6 + 4x^5 - x^4 - 6x^3 - 8x^2 + 2x + 8$$

Student A

-4	1	4	-1	-6	-8	2	8
		-4	0	4	8	0	-8
	1	0	-1	-2	0	2	0

$P(-4) = 0$

Student B

4	1	4	-1	-6	-8	2	8
		4	32	124	472	1856	7432
	1	8	31	118	464	1858	7440

$P(-4) = 7440$

***8.** (94) **Analyze** When solving $\frac{(x-4)(x+6)}{(x+2)(x-3)} \geq 0$, which intervals will use strict inequalities ($<$ or $>$) and why?

***9.** (64) **Multi-Step** As of the 2000 U.S. census, Nevada had the fastest growing population of all the states, and Connecticut had nearly the slowest (47th out of 50). The table below is based on U.S. census statistics.

	Nevada	Connecticut
Population in 2000 (to nearest thousand)	1,998,000	3,406,000
Average annual increase (1990–2000)	5.21%	0.36%

a. Write a function ($y_1 = ab^t$), where y_1 represents Nevada's population t years after 2000.

b. Write a function ($y_2 = ab^t$), where y_2 represents Connecticut's population t years after 2000.

c. Write and solve an equation to predict the year in which Nevada's population will overtake Connecticut's population.

Simplify each of the following.

10. (81) $2 \ln e^x$

11. (81) $x \cdot \ln e^3$

12. (73) **Estimate** A researcher visits a wildlife preserve and marks endangered African elephants. He marks 6 elephants and releases them. On a return visit several months later, he comes across 16 elephants, and 4 of them are marked from the previous visit. Estimate the elephant population in the preserve.

13. (77) **Surveying** A surveyor finds that the lengths between three stakes on the ground are 125 feet, 182 feet, and 211 feet. He connects the stakes with string, forming a triangle. To the nearest tenth, what is the measure of the largest angle in this triangle?

14. (79) Determine the domain and range of the function $f(x) = \begin{cases} -x, & \text{if } x < 0 \\ 4, & \text{if } 0 \le x < 10. \\ 18, & \text{if } x \ge 10 \end{cases}$

15. (80) **Research** In a study of 413 men, the mean height was 174.1 centimeters and the standard deviation was 7 centimeters. Assuming that the distribution is normal, approximate the z-score of a man whose height is 182 centimeters.

16. (80) A set of values has a mean of 13 and a standard deviation of 0.625. Find the z-score of a value of 10.25.

17. (11) **Justify** Formulate a conjecture (statement) about the degree of a polynomial that is a sum or difference of polynomials compared to the degree of the polynomials that are added or subtracted to get that sum or difference. Give examples to justify your conjecture. (Hint: Name the polynomials P_1, P_2, P_3, etc. to make it easy to refer to them.)

18. (23) **Geometry** The area of a rectangular field is 100 meters. The field is 15 meters longer than it is wide. Find the length and width of the field.

***19.** (95) **Multiple Choice** Which of the following polynomials has $P(5) = 0$?

A $P(x) = 48x^6 + 212x^5 - 1098x^4 - 5298x^3 + 1174x^2 + 22{,}350x + 18{,}900$

B $P(x) = 48x^6 + 644x^5 + 2754x^4 + 2154x^3 - 12{,}974x^2 - 30{,}750x - 18{,}900$

C $P(x) = 48x^6 + 868x^5 + 6282x^4 + 23{,}238x^3 + 46{,}274x^2 + 46{,}950x + 18{,}900$

D $P(x) = 48x^6 - 268x^5 - 818x^4 + 4282x^3 + 6254x^2 - 14{,}790x - 18{,}900$

20. (1) **Multiple Choice** Identify which property or properties of real numbers are being demonstrated.

$$(2 \cdot 9) \cdot 5 = (2 \cdot 5) \cdot 9$$

A Commutative Property of Multiplication

B Associative Property of Multiplication

C Distributive Property

D Both A and B

***21.** (95) Write a possible polynomial $P(x)$ that fits the parameters $P(0) = 0$, $P(2) = 20$, and $P(-4) = 0$.

***22.** (87) Use the properties of logarithms to expand the expression $\ln (8x)^3 + \ln e^{\left(\frac{2x}{3}\right)}$.

23. **Predict** The table below shows the sales in dollars, y, made by a gift shop x years after it opened. (13)

Years	2	5	8
Sales	$30,000	$75,000	$120,000

a. Plot the data on a coordinate grid and draw a line that fits the data.

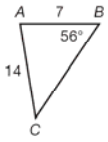
b. Use the line to predict the total sales 15 years after the shop opened.

24. **Error Analysis** A student performs the following steps while finding the roots of $y = (x^2 - 7)(3x^2 + 4) - (x^2 - 7)(8x - 3)$. What is his mistake? (76)

$$(x^2 - 7)(3x^2 + 4) - (x^2 - 7)(8x - 3) = (x^2 - 7)(3x^2 - 8x + 1)$$

***25.** Travel A driver on the Pennsylvania turnpike turned on the cruise control at milepost 30, Warrendale, and kept it on until milepost 242, Harrisburg West, exactly 4 hours later. Show how to use an arithmetic sequence to find the speed at which the car traveled during those 4 hours. (92)

26. Solve $\triangle ABC$. (71)



***27.** Write the equation for the circle centered at (3, 8) with a radius of 4. (91)

28. John saves $125 each month. Use the Distributive Property to mentally calculate the amount of money that he will save in 18 months. (1)

***29.** **Graphing Calculator** Use a graphing calculator to graph the function $y = 3 \tan(2x - 2\pi) - 4$. Identify its period, undefined values, and phase shift. (90)

***30.** Write a quadratic equation whose root is -3. (83)